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Note

On the divisibility of the cycle number by 7

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Abstract

For every even $p \geqslant 4$, there exists a nonseparable cubic graph G with p vertices, such that the cycle number of G is divisible by 7.

The subject of this note which pertains to the number of cycles in graphs, or a graph's cycle number, first grew around a question posed by Gerhard Ringel. It was noticed that for small p there existed a cubic graph with p vertices and cycle number divisible by 7. The question asked was whether there existed a cubic graph on p vertices for any even p such that its cycle number was divisible by seven. Terminology can be found in [1].

Theorem 1. *For every even $p \geqslant 4$, there exists a nonseparable cubic graph G, such that the cycle number of G is divisible by 7.*

As a preliminary step in proving the theorem, the operation of merging two graphs along an edge will be examined.

Merging of two graphs along an edge

Take two graphs, G_1 and G_2, with cycle numbers c_1 and c_2, respectively. Choose an edge (u_1, v_1) from G_1 which lies on x_1 cycles and an edge (u_2, v_2) from G_2 which lies

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on x_2 cycles. Delete the edges (u_1, v_1) and (u_2, v_2) and join u_1 to u_2 and v_1 to v_2 with two new edges. Subdivide these edges, calling the new vertices s and t and join these two new vertices with the new edges (s, t). See Fig. 1.

Note that the c_1 and c_2 cycles of G_1 and G_2 are still retained. The newly formed graph G has $c_1 + c_2$ cycles and all those new cycles which have been formed which must include the paths $u_1 - s - u_2$ and $v_1 - t - v_2$. These are the only additional cycles that G can contain.

There are x_1 paths from u_1 to v_1 in $G_1 - (u_1, v_1)$ and x_2 paths from u_2 to v_2 in $G_2 - (u_2, v_2)$. Therefore, there are $x_1 \cdot x_2$ cycles which include the paths $u_1 - s - u_2$ and $v_1 - t - v_2$. This results in a total of $c_1 + c_2 + x_1 \cdot x_2$ cycles in G.

In addition, the new edge (s, t) lies on $x_1 + x_2$ cycles and the edges (u_1, s), (v_1, t) on $x_1 + x_1 \cdot x_2$ cycles and (u_2, s), (v_2, t) on $x_2 + x_1 \cdot x_2$ cycles.

Proof of Theorem 1. Choose an edge, say e, of a graph G which lies on x cycles, and say G has c cycles. As in the above construction, merge G and the graph K_4 along the edge e of G and any edge of K_4. Call two of the newly formed edges f and h. See Fig. 2. Since K_4 has 7 cycles and each of its edges lies on 4 cycles, the new graph now has $c + 7 + 4x$ cycles. Therefore, if G is such a graph where $c \equiv 0 \pmod 7$ and $x \equiv 0 \pmod 7$, then the new graph will have a cycle number also divisible by 7. Such graphs are shown in Fig. 3 for $p = 6$, 8, and 10.

With each merging of such a G and K_4, the new graph will always have an edge which lies on a number of cycles divisible by 7. This can be seen by considering the edge f, as above, with the edge on $x + 4x = 5x$ cycles, where $x = 0 \pmod 7$.

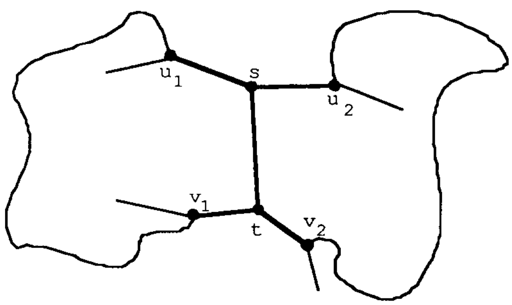


Fig. 1.

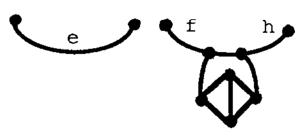


Fig. 2.

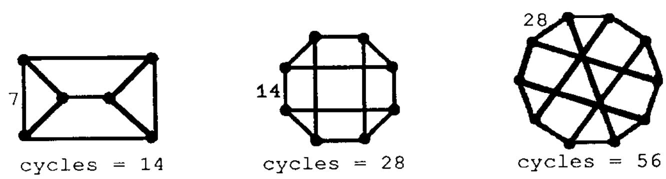


Fig. 3.

Since 6 new vertices are added to G with each new merge, a graph with any p vertices may be obtained from one of the three in Fig. 3. Fig. 4 shows the construction for $p=6+6k$, $8+6k$, and $10+6k$ for $k=1$. □

The above construction can also be utilized to find a particular cycle number with any order modulo 7 of a cubic graph with p vertices.

Theorem 2. *There is a nonseparable cubic graph on p vertices whose cycle number is congruent to k* (mod 7) *where* $k=0$ *when* $p \geqslant 4$, $k=1$ *when* $p \geqslant 6$, $k=5$ *when* $p \geqslant 8$, *and* $k=2,3,4,6$ *when* $p \geqslant 10$.

Proof. $k=1$. The construction is as above. The graph in the first illustration in Fig. 3 shows the traingular prism with 6 vertices, 14 cycles and an edge, say e, which lies on 7 cycles. Notice that this is the cubic graph of smallest order with the property of having a cycle number divisible by 7 and an edge which lies on a number of cycles divisible 7. If this is merged with cubic graphs with cycle number $c \equiv 1$ (mod 7), the number of cycles in the new graph will be $c+14+7x$, where x is the number of cycles on which the subdivided edge of the chosen graph lies. Again, as pointed out in the explanation of the construction, one of the new edges created by the merge will lie on a number of edges divisible by 7. Fig. 5 shows the cubic graphs with 6, 8, 10, and 12 vertices and $c \equiv 1$ (mod 7). As an example, the merging of the triangular prism and the graph in the first illustration of Fig. 5 yields a cubic graph with 14 vertices and $15+14+7 \cdot 8=85$ cycles.

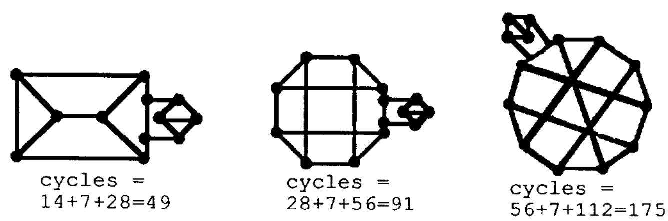


Fig. 4.

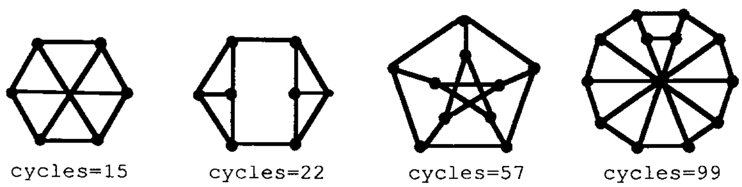
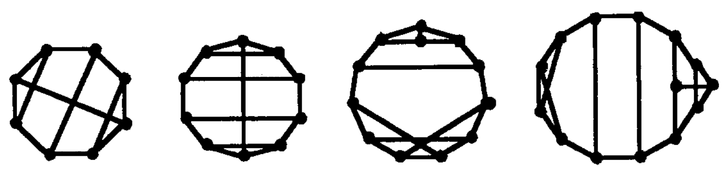


Fig. 5.

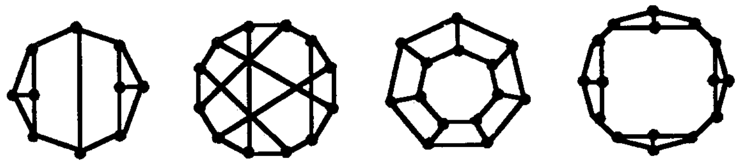
The cases for $k=5, 2$, and 4 are done in the same manner and the pertinent graphs are shown below in Figs. 6, 7, and 8.

$k=3$. The construction for this case will be done on the graphs obtained from Theorem 1. Find such a graph with $p-4$ vertices, cycle number c, and find an edge in this graph which lies on $x \equiv 0 \pmod 7$ cycles. Subdivide this edge three times and adjoin the graph which is depicted by the first illustration in Fig. 9. This will result in



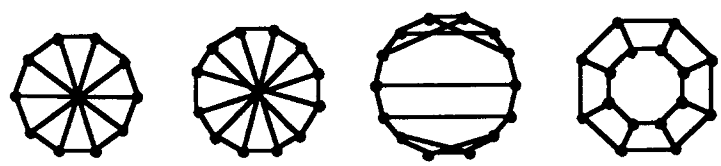
cycles=26 cycles=47 cycles=54 cycles=61

Fig. 6.



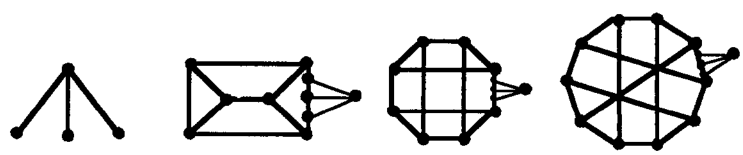
cycles=30 cycles=107 cycles=170 cycles=268

Fig. 7.



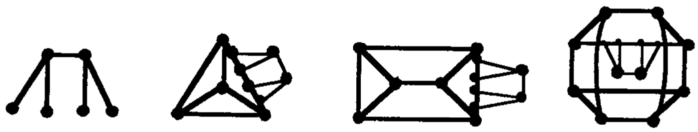
cycles=53 cycles=95 cycles=67 cycles=312

Fig. 8.



cycles=14+3 +3·7=38 cycles=28+3 +3·16 = 79 cycles=56+3 +3·28 =143

Fig. 9.



cycles=7+6 +7·4 = 41 cycles=14+6 +7·7 = 69 cycles=28+6 +7·16 = 146

Fig. 10.

$c+3+3x$ cycles which is $\equiv 3 \pmod 7$. The graphs for $p=10$, 12, and 14 are also shown in Fig. 9.

$k=6$. The construction is similar to the case for $k=3$, except that the edge of a graph from Theorem 1 is subdivided four times and the graph in the first illustration in Fig. 10 is joined. This will result in $c+6+7x$ cycles which is $\equiv 6 \pmod 7$. The graphs for $p=10$, 12, and 14 are also shown in Fig. 10. Note that any edge may be chosen to subdivide. □

References

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs (Wadsworth, New York, 1979).

[2] F.C. Bussemaker, D.M. Cobeljić, D.M. Cvetbović and J.J. Seidel, Computer Investigation of Cubic Graphs. T.H.-Report 76-WSK-01, Technological University Eindhoven Netherlands, 1976.